

Ascent Sequences and 3-Nonnesting Set Partitions

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Abstract. A sequence $x = x_1x_2 \dots x_n$ is said to be an ascent sequence of length n if it satisfies $x_1 = 0$ and $0 \leq x_i \leq \text{asc}(x_1x_2 \dots x_{i-1}) + 1$ for all $2 \leq i \leq n$, where $\text{asc}(x_1x_2 \dots x_{i-1})$ is the number of ascents in the sequence $x_1x_2 \dots x_{i-1}$. Recently, Duncan and Steingrímsson proposed the conjecture that 210-avoiding ascent sequences of length n are equinumerous with 3-nonnesting set partitions of $\{1, 2, \dots, n\}$. In this paper, we confirm this conjecture by showing that 210-avoiding ascent sequences of length n are in bijection with 3-nonnesting set partitions of $\{1, 2, \dots, n\}$ via an intermediate structure of growth diagrams for 01-fillings of Ferrers shapes.

KEY WORDS: ascent sequence, pattern avoiding, 3-nonnesting set partition, growth diagram, 01-filling of Ferrers shape.

AMS MATHEMATICAL SUBJECT CLASSIFICATIONS: 05A05, 05C30.

1 Introduction

The objective of this paper is to establish a bijection between 210-avoiding ascent sequences of length n and 3-nonnesting set partitions of $\{1, 2, \dots, n\}$. Let us give an overview of the notation and terminology.

Given a sequence of integers $x = x_1x_2 \dots x_n$, we say that the sequence x has an ascent at position i if $x_i < x_{i+1}$. The number of ascents of x is denoted by $\text{asc}(x)$. A sequence $x = x_1x_2 \dots x_n$ is said to be an *ascent sequence of length n* if it satisfies $x_1 = 0$ and $0 \leq x_i \leq \text{asc}(x_1x_2 \dots x_{i-1}) + 1$ for all $2 \leq i \leq n$. Ascent sequences were introduced by Bousquet-Mélou et al. [1] in their study of $(2+2)$ -free posets. Ascent sequences are closely connected to many other combinatorial structures. Bousquet-Mélou et al. [1] constructed bijections between unlabeled $(2+2)$ -free posets and ascent sequences, between ascent sequences and permutations avoiding a certain pattern, between unlabeled $(2+2)$ -free posets and a class of involutions introduced by Stoimenow [7]. Dukes and Parviainen [3] established a bijection between ascent sequences and upper triangular matrices with non-negative integer entries such that all rows and columns contain at least one non-zero entry. We

call an ascent sequence with no two consecutive equal entries a *primitive* ascent sequence. For example, the sequence 0120234603 is a primitive ascent sequence of length 10.

Analogous to pattern avoidance on permutations, Duncan and Steingrímsson [4] initiated the study of ascent sequences avoiding certain patterns. For ascent sequences, a *pattern* is a word on nonnegative integers $\{0, 1, \dots, k\}$, where each element appears at least once. Given an ascent sequence $x = x_1x_2 \dots x_n$ and a pattern $\tau = \tau_1\tau_2 \dots \tau_k$, we say that a subsequence $x_{i_1}x_{i_2} \dots x_{i_k}$ of x is a pattern of τ if it is order-isomorphic to τ . If x contains no subsequence of pattern τ , then we say that x is τ -*avoiding*. Denote by $\mathcal{A}_n(\tau)$ and $\mathcal{PA}_n(\tau)$ the set of τ -avoiding ordinary and primitive ascent sequences, respectively. Duncan and Steingrímsson [4] proved that $|\mathcal{A}_n(\tau)| = C_n$, the n th Catalan number, for any $\tau = 101, 0101$ or 021 . Moreover, they proposed the following conjecture.

Conjecture 1.1 *210-avoiding ascent sequences of length n are equinumerous with 3-nonnesting (3-noncrossing) set partitions of $\{1, 2, \dots, n\}$.*

Note that Mansour and Shattuck [6] recently derived two recurrence relations on the generating function for 210-avoiding ascent sequences. However, as remarked by Mansour and Shattuck, the conjecture is still open.

Recall that a set partition P of $[n] = \{1, 2, \dots, n\}$ can be represented by a diagram with vertices drawn on a horizontal line in increasing order. For a block B of P , we write the elements of B in increasing order. Suppose that $B = \{i_1, i_2, \dots, i_k\}$. Then we draw an arc from i_1 to i_2 , an arc from i_2 to i_3 , and so on. Such a diagram is called the linear representation of P , see Figure 1 for an example. We say that k arcs $(i_1, j_1), (i_2, j_2), \dots, (i_k, j_k)$ form a k -crossing if $i_1 < i_2 < \dots < i_k < j_1 < j_2 < \dots < j_k$. A partition without any k -crossing is said to be k -*noncrossing*. Similarly, a k -nesting is a set of k arcs $(i_1, j_1), (i_2, j_2), \dots, (i_k, j_k)$ such that $i_1 < i_2 < \dots < i_k < j_k < \dots < j_2 < j_1$. A set partition without any k -nesting is said to be k -*nonnesting*. Chen et al. [2] proved that k -nonnesting set partitions of $[n]$ are equinumerous with k -noncrossing set partitions of $[n]$ bijectively using vacillating tableaux as an intermediate object.

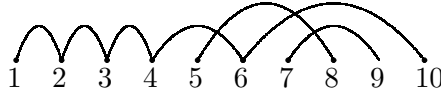


Figure 1: The linear representation of a set partition $\pi = \{\{1, 2, 3, 4, 6, 10\}, \{5, 8\}, \{7, 9\}\}$.

In this paper, we aim to establish a bijection between 210-avoiding ascent sequences of length n and 3-nonnesting set partitions of $[n]$ via an intermediate structure of growth diagrams for 01-fillings of Ferrers shapes described in [5] and [8].

2 Growth diagrams for 01-fillings of Ferrers shapes

Our bijection between 210-avoiding ascent sequences and 3-nonnesting set partitions will be accomplished by the growth diagram for 01-fillings of Ferrers shapes [5]. Before we describe the growth diagram for 01-fillings of Ferrers shapes, we give an overview of the notation and terminology.

A *partition* λ of a positive integer n is defined to be a sequence $(\lambda_1, \lambda_2, \dots, \lambda_m)$ of nonnegative integers such that $\lambda_1 + \lambda_2 + \dots + \lambda_m = n$ and $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_m$. The empty partition is denoted by \emptyset . For the sake of convenience, we identify a partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_m)$ with the infinite sequence $(\lambda_1, \lambda_2, \dots, \lambda_m, 0, 0, \dots)$, that is, the sequence obtained from λ by appending infinitely many 0's. Given a partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_m)$, its *Ferrers diagram* is the left-justified array of $\lambda_1 + \lambda_2 + \dots + \lambda_m$ squares with λ_1 squares in the first row, λ_2 squares in the second row, and so on. The *partial order* \subseteq on partitions is defined by the containment of their Ferrers diagrams. The *conjugate* of a partition λ is the partition $(\lambda'_1, \dots, \lambda'_{\lambda_1})$ where λ'_j is the length of the j th column in the Ferrers diagram of λ .

A Ferrers shape is a Ferrers diagrams in French notation which has straight left side, straight bottom side and supports a descending staircase. We can also encode a Ferrers shape F by sequences of D 's and R 's by tracing the right/up boundary of F from top-left to bottom-right and writing D (resp. R) whenever we encounter a down-step (resp. right-step). For example, the Ferrers shape in Figure 2 can be represented by $RRDDRRDRD$.

A 01-*filling* of a Ferrers shape is obtained by filling each cells of F with 1's and 0's, see Figure 2 for an example, where we present 1's by \bullet and suppress the 0's. A *NE-chain* of a 01-filling is a sequence of 1's such that any 1 is strictly above and weakly to the right of the preceding 1 in the sequence. A *SE-chain* of a 01-filling is a sequence of 1's such that any 1 is weakly below and strictly to the right of the preceding 1 in the sequence. The *length* of a *NE-chain* or a *SE-chain* is defined to the number of 1's in the chain.

The *growth diagram* for a 01-filling of Ferrers shape F is obtained by labelling the corners of all the squares in F by partitions in such a way that the partition assigned to any corner is either equal to the partition to its left or is obtained

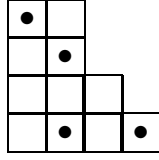


Figure 2: Example of a 01-filling of a Ferrers shape.

from it by adding a *horizontal strip*, that is, by a set of squares no two of which are in the same column, and the partition assigned to any corner either equals the partition below it or is obtained from this partition by adding a *vertical strip*, that is, by a set of squares no two of which are in the same row. We start by assigning the partition \emptyset to each corner on the left and bottom edges of F . Then assign the partitions to the other corners inductively by applying the following forward algorithm. Consider the cell in Figure 3, filled by $m = 0$ or $m = 1$ and labeled by the partitions ρ, μ, v , where $\rho \subseteq \mu$ and $\rho \subseteq v$, μ and ρ differ by a horizontal strip, and v and ρ differ by a vertical strip. Then λ is determined by the following algorithm.

- (F0) Set $CARRY := m$ and $i := 1$.
- (F1) Set $\lambda_i = \max\{\mu_i + CARRY, v_i\}$.
- (F2) If $\lambda_i = 0$, then stop. The output of the algorithm is $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_{i-1})$. If not, then set $CARRY := \min\{\mu_i + CARRY, v_i\} - \rho_i$ and $i := i + 1$. GO to (F1).

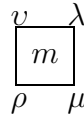


Figure 3: A cell filled with m .

Conversely, given a labelling of the corners along the right/up border of a Ferrer shape F , one can reconstruct the labels of the other corners and a 01-filling of F inductively by applying the following backward algorithm. Consider the cell in Figure 3 labelled by the partitions μ, v, λ , where $\mu \subseteq \lambda$ and $v \subseteq \lambda$, where λ and μ differ by a vertical strip, and λ and v differ by a horizontal strip, the backward algorithm works in the following way.

- (B0) Set $i := \max\{j : \lambda_j \text{ is positive}\}$ and $CARRY := 0$.

- (B1) Set $\rho_i = \min\{\mu_i, v_i - CARRY\}$.
- (B2) Set $CARRY := \lambda_i - \max\{\mu_i, v_i - CARRY\}$ and $i := i - 1$. If $i = 0$, then stop. The output of the algorithm is $\rho = (\rho_1, \rho_2, \dots)$ and $m = CARRY$. If not, go to (B1).

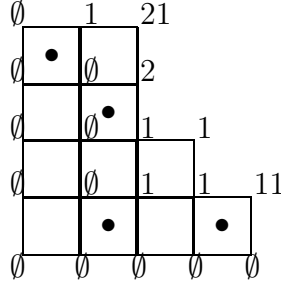


Figure 4: The growth diagram for the 01-filling in Figure 2.

Theorem 2.1 (See [5], Theorem 9) *Let F be a Ferrers shape given by $D - R$ -sequence $w = w_1 w_2 \dots w_k$. Then 01-fillings of F are in bijection with sequences $(\emptyset = \lambda^0, \lambda^1, \dots, \lambda^k = \emptyset)$ where λ^{i+1} is obtained from λ^i by doing nothing (i.e., $\lambda^{i+1} = \lambda^i$) or adding a horizontal strip if $w_i = R$, whereas λ^i is obtained from λ^{i-1} by doing nothing or deleting a vertical strip if $w_i = D$.*

Theorem 2.2 (See [5], Theorem 10) *Given a diagram with empty partitions labelling all the corners along the left side and the bottom side of the Ferrers shape, suppose that the corner c is labelled by the partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_l)$. Then λ_1 is the length of the longest NE-chain in the rectangular region to the left and below of c , and λ'_1 is the length of the longest SE-chain in the same rectangular region.*

Lemma 2.3 *Given a 01-filling of Ferrers shape F given by $D - R$ -sequence $w = w_1 w_2 \dots w_k$, let $(\emptyset = \lambda^0, \lambda^1, \dots, \lambda^k = \emptyset)$ be its corresponding sequence. Then λ^i is obtained from λ^{i-1} by adding a square if and only if there is exactly one 1 in the column of the cells of F below the corners labeled by λ^{i-1} and λ^i , and λ^i is obtained from λ^{i-1} by deleting a square if and only if there is exactly one 1 in the row of the cells of F to the left of the corners labeled by λ^{i-1} and λ^i .*

Proof. Let c be a cell of F illustrated as Figure 3, which is filled by $m = 0$ or $m = 1$ and labeled by the partitions ρ, μ, v , where $\rho \subseteq \mu$ and $\rho \subseteq v$, μ and ρ differ by at most one square, and v and ρ differ by at most one square. In order to prove the lemma, it suffices to show that λ has the following properties.

- (a) If $m = 0$ and $\rho = v$, then $\lambda = \mu$.
- (b) If $m = 1$ and $\rho = v$, then λ and μ differ by a square.
- (c) If $m = 0$ and ρ and v differ by a square, then λ and μ differ by a square.
- (d) If $m = 0$ and $\rho = \mu$, then $\lambda = v$.
- (e) If $m = 1$ and $\rho = \mu$, then λ and v differ by a square.
- (f) If $m = 0$ and ρ and μ differ by a square, then λ and v differ by a square.

Here we reformulate the forward algorithm in a slightly different, but of course equivalent fashion. Let $a_1 = m$ and $\lambda_1 = \max\{\mu_1 + a_1, v_1\}$. Suppose that $\lambda_1, \lambda_2, \dots, \lambda_i$ and a_1, a_2, \dots, a_i are already determined. If $\lambda_i = 0$, then let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_{i-1})$. Otherwise, let $a_{i+1} = \min\{\mu_i + a_i, v_i\} - \rho_i$ and $\lambda_{i+1} = \max\{\mu_{i+1} + a_{i+1}, v_{i+1}\}$. Repeat the above procedure until we get $\lambda_k = 0$ and let $\lambda = \{\lambda_1, \lambda_2, \dots, \lambda_{k-1}\}$.

If $m = 0$ and $\rho = v$, then we have $a_i = 0$ for all $i \geq 1$ since $\mu_i \geq \rho_i$ for all $i \geq 1$. This implies that $\lambda = \mu$.

If $m = 1$ and $\rho = v$, then we have $a_1 = 1$ and $a_i = 0$ for all $i \geq 2$. According to the forward algorithm, we have $\lambda_1 = \max\{\mu_1 + 1, v_1\} = \mu_1 + 1$ and $\lambda_i = \max\{\mu_i + a_i, v_i\} = \mu_i$ for all $i \geq 2$. This yields that λ and μ differ by a square.

If $m = 0$ and ρ and v differ by a square, then choose the integer j such that $v_j = \rho_j + 1$. In this case we have $a_i = 0$ for all $i \leq j$ and $\lambda_i = \max\{\mu_i + a_i, v_i\} = \mu_i$ for all $i < j$. Moreover, for all $i \geq j + 1$, we have $a_{i+1} = \min\{\mu_i + a_i, v_i\} - \rho_i = 0$ and $\lambda_{i+1} = \max\{\mu_{i+1} + a_{i+1}, v_{i+1}\} = \mu_{i+1}$.

In order to verify property (c), it remains to show that we have either $\mu_j = \lambda_j$ and $\lambda_{j+1} = \mu_{j+1} + 1$, or $\mu_{j+1} = \lambda_{j+1}$ and $\lambda_j = \mu_j + 1$. We have two cases. If $\mu_j > \rho_j$, then we have $\lambda_j = \max\{\mu_j + a_j, v_j\} = \max\{\mu_j, v_j\} = \mu_j$, $a_{j+1} = \min\{\mu_j, v_j\} - \rho_j = 1$ and $\lambda_{j+1} = \max\{\mu_{j+1} + a_{j+1}, v_{j+1}\} = \mu_{j+1} + 1$ since $v_{j+1} = \rho_{j+1} \leq \mu_{j+1}$. If $\mu_j = \rho_j$, then we have $\lambda_j = \max\{\mu_j + a_j, v_j\} = \max\{\mu_j, v_j\} = v_j = \mu_j + 1$, $a_{j+1} = \min\{\mu_j, v_j\} - \rho_j = 0$ and $\lambda_{j+1} = \max\{\mu_{j+1} + a_{j+1}, v_{j+1}\} = \mu_{j+1}$ since $v_{j+1} = \rho_{j+1} \leq \mu_{j+1}$. So far, we have reached the conclusion that λ has the properties (a) – (c). By the same reasoning as in the proofs of (a) – (c), we can verify properties (d) – (f). The details are omitted. This completes the proof. ■

In this paper, we are mainly concerned with 01-fillings of triangular shape. Let Δ_n be the triangular shape with n cells in the bottom row, $n - 1$ cells in the row above, etc., and 1 cell in the topmost row. Combining Theorems 2.1, 2.2 and Lemma 2.3, we have the following theorems.

Theorem 2.4 *01-fillings of Δ_n with the property that every row contains exactly one 1 are in bijection with sequences $(\emptyset = \lambda^0, \lambda^1, \dots, \lambda^{2n} = \emptyset)$ where λ^{2i+1} is obtained from λ^{2i} by doing nothing or adding a horizontal strip, whereas λ^{2i+2} is obtained from λ^{2i+1} by deleting a square.*

Theorem 2.5 *01-fillings of Δ_n with the property that every row and every column contains at most one 1 are in bijection with sequences $(\emptyset = \lambda^0, \lambda^1, \dots, \lambda^{2n} = \emptyset)$ where λ^{2i+1} is obtained from λ^{2i} by doing nothing or adding a square, whereas λ^{2i+2} is obtained from λ^{2i+1} by doing nothing or deleting a square.*

Theorem 2.6 *01-fillings of Δ_n with the property every row contains exactly one 1 and there is no NE-chain of length $k+1$ are in bijection with sequences $(\emptyset = \lambda^0, \lambda^1, \dots, \lambda^{2n} = \emptyset)$ where the most number of columns of any λ^i is at most k , and λ^{2i+1} is obtained from λ^{2i} by doing nothing or adding a horizontal strip, whereas λ^{2i+2} is obtained from λ^{2i+1} by deleting a square.*

Theorem 2.7 *01-fillings of Δ_n with the property that every row and every column contains at most one 1 and there is no NE-chain of length $k+1$ are in bijection with sequences $(\emptyset = \lambda^0, \lambda^1, \dots, \lambda^{2n} = \emptyset)$ where the most number of columns of any λ^i is at most k , and λ^{2i+1} is obtained from λ^{2i} by doing nothing or adding a square, whereas λ^{2i+2} is obtained from λ^{2i+1} by doing nothing or deleting a square.*

Recall that there is a bijection between set partitions of $[n]$ and 01-fillings of Δ_{n-1} in which every row and every column contains at most one 1. Given a set partition π of $[n]$, we can get a 01-filling of Δ_{n-1} by putting a 1 in the i th column and j th row from above (where we number rows such that the row consisting of $j-1$ cells is numbered j), whenever (i, j) is an arc in its linear representation. The 01-filling corresponding to the set partition $\pi = \{\{1, 2, 3, 4, 6, 10\}, \{5, 8\}, \{7, 9\}\}$ is shown in Figure 5. Moreover, a k -crossing of a set partition corresponds to a SE -chain of length k in the filling, while a k -nesting corresponds to a NE -chain of length k . Thus the following theorem follows immediately.

Theorem 2.8 *k -nonnesting set partitions of $[n+1]$ are in bijection with 01-fillings of Δ_n with the property that every row and every column contains at most one 1 and there is no NE-chain of length k .*

Denote by \mathcal{V}_n the set of sequences $(\emptyset = \lambda^0, \lambda^1, \dots, \lambda^{2n} = \emptyset)$ where the most number of columns of any λ^i is at most 2, and λ^{2i+1} is obtained from λ^{2i} by

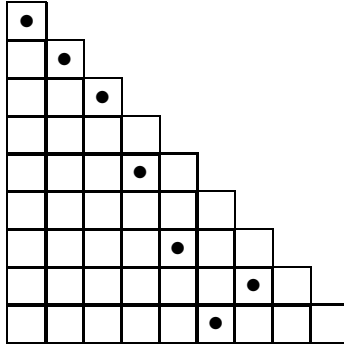


Figure 5: A set partition $\pi = \{\{1, 2, 3, 4, 6, 10\}, \{5, 8\}, \{7, 9\}\}$ and its corresponding 01-filling.

doing nothing or adding a square, whereas λ^{2i+2} is obtained from λ^{2i+1} by doing nothing or deleting a square.

In view of Theorems 2.7 and 2.8, in order to prove Conjecture 1.1, it suffices to establish a bijection between the set $\mathcal{A}_{n+1}(210)$ and the set \mathcal{V}_n . In the next section, we will provide such a bijection.

3 Proof of the conjecture

In this section, we will establish a bijection between the set $\mathcal{A}_{n+1}(210)$ and the set \mathcal{V}_n . To this end, we first construct a bijection between 210-avoiding primitive ascent sequences and a certain class of 01-fillings of triangular shape.

Denote by $\mathcal{N}(\Delta_n)$ the set of all 01-fillings of Δ_n with the property that every row contains exactly one 1 and there is no NE-chain of length 3. In this section, a 01-filling in $\mathcal{N}(\Delta_n)$ will be identified with a sequence $\{(1, a_1), (2, a_2), \dots, (n, a_n)\}$, where $1 \leq a_i \leq i$ and $a_i = k$ if and only if there is a 1 in the i th row and k th column (where we number columns from left to right and rows such that the row consisting of j cells is numbered j). For example, the filling in figure 6 is identified with

$$\{(1, 1), (2, 2), (3, 3), (4, 1), (5, 4), (6, 4), (7, 5), (8, 7), (9, 6)\}.$$

Let $n \geq 1$. We aim to describe a map ϕ from the set $\mathcal{PA}_{n+1}(210)$ to the set $\mathcal{N}(\Delta_n)$. Let $x = x_1 x_2 \dots x_{n+1} \in \mathcal{PA}_{n+1}(210)$. Define $\phi(x) = \{(1, a_1), (2, a_2), \dots, (n, a_n)\}$ where $a_i = i + x_{i+1} - \text{asc}(x_1 x_2 \dots x_{i+1})$ for all $i = 1, 2, \dots, n$. For example, let

$x = 012340415 \in \mathcal{PA}_9(210)$. Then we have

$$\phi(x) = \{(1, 1), (2, 2), (3, 3), (4, 4), (5, 1), (6, 5), (7, 3), (8, 7)\}.$$

Lemma 3.1 *Let $x = x_1x_2 \dots x_{n+1} \in \mathcal{PA}_{n+1}(210)$ and $\phi(x) = \{(1, a_1), (2, a_2), \dots, (n, a_n)\}$. Suppose that $a_i \geq a_j$ with $i < j$. Then we have $x_{i+1} > x_{j+1}$.*

Proof. Recall that $a_i = i + x_{i+1} - \text{asc}(x_1x_2 \dots x_{i+1})$ and $a_j = j + x_{j+1} - \text{asc}(x_1x_2 \dots x_{j+1})$. Since $a_i \geq a_j$, we have

$$x_{i+1} - x_{j+1} \geq j - i - (\text{asc}(x_1x_2 \dots x_{j+1}) - \text{asc}(x_1x_2 \dots x_{i+1})) \geq 0.$$

If $x_{i+2} < x_{i+1}$, then we have $\text{asc}(x_1x_2 \dots x_{j+1}) \leq \text{asc}(x_1x_2 \dots x_{i+1}) + j - i - 1$. This yields that

$$x_{i+1} - x_{j+1} \geq j - i - (\text{asc}(x_1x_2 \dots x_{j+1}) - \text{asc}(x_1x_2 \dots x_{i+1})) \geq 1.$$

Thus we deduce that $x_{i+1} > x_{j+1}$.

If $x_{i+2} > x_{i+1}$, then we have $\text{asc}(x_1x_2 \dots x_{j+1}) = \text{asc}(x_1x_2 \dots x_{i+1}) + 1 + \text{asc}(x_{i+2} \dots x_{j+1})$. If $\text{asc}(x_{i+2} \dots x_{j+1}) = j - i - 1$, then we have $x_{i+2} < \dots < x_{j+1}$. Thus it follows that $x_{j+1} > x_{i+1}$. This contradicts with the fact that $x_{i+1} - x_{j+1} \geq 0$. Hence we have $\text{asc}(x_{i+2} \dots x_{j+1}) < j - i - 1$. It follows that

$$\begin{aligned} \text{asc}(x_1x_2 \dots x_{j+1}) &= \text{asc}(x_1x_2 \dots x_{i+1}) + 1 + \text{asc}(x_{i+2} \dots x_{j+1}) \\ &< \text{asc}(x_1x_2 \dots x_{i+1}) + j - i. \end{aligned}$$

This implies that $x_{i+1} - x_{j+1} \geq 1$. This completes the proof. \blacksquare

Theorem 3.2 *For $n \geq 1$, the map ϕ is well defined, that is, for any ascent sequence $x \in \mathcal{PA}_{n+1}(210)$, we have $\phi(x) \in \mathcal{N}(\Delta_n)$.*

Proof. Let $x = x_1x_2 \dots x_{n+1} \in \mathcal{PA}_{n+1}(210)$ and $\phi(x) = \{(1, a_1), (2, a_2), \dots, (n, a_n)\}$. First it is necessary to prove that $a_i \leq i$. According to the definition of ascent sequences, we have $x_{i+1} \leq \text{asc}(x_1x_2 \dots x_i) + 1$.

If $x_{i+1} = \text{asc}(x_1x_2 \dots x_i) + 1$, then we have $\text{asc}(x_1x_2 \dots x_{i+1}) = \text{asc}(x_1x_2 \dots x_i) + 1$. This yields that $x_{i+1} = \text{asc}(x_1x_2 \dots x_{i+1})$, which implies that $a_i = i$.

If $x_{i+1} \leq \text{asc}(x_1x_2 \dots x_i)$, then it follows that

$$x_{i+1} \leq \text{asc}(x_1x_2 \dots x_i) \leq \text{asc}(x_1x_2 \dots x_{i+1}).$$

Recall that $a_i = i + x_{i+1} - \text{asc}(x_1x_2 \dots x_{i+1})$. Thus we deduce that $a_i \leq i$.

Next we aim to show that $a_i \geq 1$. There are two cases. If $\text{asc}(x_1 x_2 \dots x_{i+1}) = i$, then we have $0 = x_1 < x_2 < \dots < x_{i+1}$. In this case, we have

$$a_i = i + x_{i+1} - \text{asc}(x_1 x_2 \dots x_{i+1}) = x_{i+1} \geq 1.$$

If $\text{asc}(x_1 x_2 \dots x_{i+1}) < i$, then we have

$$a_i = i + x_{i+1} - \text{asc}(x_1 x_2 \dots x_{i+1}) \geq x_{i+1} + 1 \geq 1.$$

Thus we have reached the conclusion that $1 \leq a_i \leq i$. This implies that $\phi(x)$ is a 01-filling of Δ_n with the property that every row contains exactly one 1.

It remains to show that there is no NE-chain of length 3 in $\phi(x)$. Suppose that the squares (i, a_i) , (j, a_j) and (k, a_k) form a NE-chain of length 3, where $i < j < k$. This means that $a_i \geq a_j \geq a_k$. By Lemma 3.1, we have $x_{i+1} > x_{j+1} > x_{k+1}$, which is a 210 pattern. This leads to a contradiction with the fact that x is 210-avoiding. Hence, we deduce that there is no NE-chain of length 3 in $\phi(x)$. This completes the proof. \blacksquare

In order to show that the map ϕ is a bijection, we proceed to describe a map ϕ' from the set $\mathcal{N}(\Delta_n)$ to the set $PA_{n+1}(210)$ for $n \geq 1$. Given a 01-filling $F \in \mathcal{N}(\Delta_n)$, we wish to recover a 210-avoiding primitive ascent sequence of length $n + 1$. Let $F = \{(1, a_1), (2, a_2), \dots, (n, a_n)\}$. Define $\phi'(F) = (x_1, x_2, \dots, x_{n+1})$ inductively as follows:

- $x_1 = 0$ and $x_2 = 1$;
- if $a_{i-1} < a_i$, then $x_{i+1} = \text{asc}(x_1 x_2 \dots x_i) + 1 + a_i - i$ for all $2 \leq i \leq n$;
- if $a_{i-1} \geq a_i$, then $x_{i+1} = \text{asc}(x_1 x_2 \dots x_i) + a_i - i$ for all $2 \leq i \leq n$.

For example, let $F = \{(1, 1), (2, 2), (3, 3), (4, 4), (5, 1), (6, 5), (7, 3), (8, 7)\} \in \mathcal{N}(\Delta_8)$. Then we have $\phi'(F) = 012340415$.

Lemma 3.3 *For any 01-filling $F = \{(1, a_1), (2, a_2), \dots, (n, a_n)\} \in \mathcal{N}(\Delta_n)$, the sequence $\phi'(F) = x_1 x_2 \dots x_{n+1}$ has the property that*

$$x_{i+1} = \text{asc}(x_1 x_2 \dots x_{i+1}) + a_i - i$$

for all $i \geq 1$. Furthermore, we have $x_i > x_{i+1}$ if $a_{i-1} \geq a_i$, whereas $x_i < x_{i+1}$ if $a_{i-1} < a_i$.

Proof. We proceed by induction on i . It is easy to check that the statement holds for $i = 1$. Assume that $x_i = \text{asc}(x_1x_2 \dots x_i) + a_{i-1} - (i-1)$. Now we shall show that $x_{i+1} = \text{asc}(x_1x_2 \dots x_{i+1}) + a_i - i$. We consider two cases.

If $a_{i-1} < a_i$, then we have $x_{i+1} = \text{asc}(x_1x_2 \dots x_i) + 1 + a_i - i$. In this case, we have $x_{i+1} > \text{asc}(x_1x_2 \dots x_i) + a_{i-1} - (i-1) = x_i$. This implies that $x_{i+1} > x_i$ and $\text{asc}(x_1x_2 \dots x_{i+1}) = \text{asc}(x_1x_2 \dots x_i) + 1$. Thus we deduce that $x_{i+1} = \text{asc}(x_1x_2 \dots x_{i+1}) + a_i - i$ and $x_i < x_{i+1}$.

If $a_{i-1} \geq a_i$, then we have $x_{i+1} = \text{asc}(x_1x_2 \dots x_i) + a_i - i$. It follows that $x_i = \text{asc}(x_1x_2 \dots x_i) + a_{i-1} - (i-1) \geq \text{asc}(x_1x_2 \dots x_i) + 1 + a_i - i = x_{i+1} + 1$. This implies that $x_i > x_{i+1}$ and $\text{asc}(x_1x_2 \dots x_{i+1}) = \text{asc}(x_1x_2 \dots x_i)$. Thus we have $x_{i+1} = \text{asc}(x_1x_2 \dots x_{i+1}) + a_i - i$ and $x_i > x_{i+1}$. This completes the proof. ■

Theorem 3.4 *Let $n \geq 1$. The map ϕ' is well defined, that is, for any 01-filling $F \in \mathcal{N}(\Delta_n)$, we have $\phi'(F) \in \mathcal{PA}_{n+1}(210)$.*

Proof. Let $F = \{(1, a_1), (2, a_2), \dots, (n, a_n)\}$ and $\phi'(F) = x_1x_2 \dots x_{n+1}$. Since $a_i \leq i$, we have $x_{i+1} \leq \text{asc}(x_1x_2 \dots x_i) + 1$ according to the definition of the map ϕ' . We next prove that $x_i \geq 0$ by induction on i . It is apparent that $x_1 \geq 0$ and $x_2 \geq 0$. Assume that $x_j \geq 0$ for all $1 \leq j \leq i$. Now we proceed to show that $x_{i+1} \geq 0$. We have two cases. If $a_{i-1} < a_i$, then we have

$$\begin{aligned} x_{i+1} &= \text{asc}(x_1x_2 \dots x_i) + 1 + a_i - i \\ &> \text{asc}(x_1x_2 \dots x_i) + a_{i-1} - (i-1). \end{aligned}$$

By Lemma 3.3, we have $x_i = \text{asc}(x_1x_2 \dots x_i) + a_{i-1} - (i-1)$. Thus it follows that $x_{i+1} > x_i \geq 0$.

If $a_{i-1} \geq a_i$, then we have $x_{i+1} = \text{asc}(x_1x_2 \dots x_i) + a_i - i$ according to the definition of ϕ' . If $\text{asc}(x_1x_2 \dots x_i) = i-1$, then we have $x_{i+1} \geq 0$ since $a_i \geq 1$. Suppose that $\text{asc}(x_1x_2 \dots x_i) < i-1$. By Lemma 3.3, there exists integer j such that $j \leq i-1$ and $a_{j-1} \geq a_j$. Let k be the largest such integer. Hence we have

$$\text{asc}(x_1x_2 \dots x_i) = \text{asc}(x_1x_2 \dots x_{k+1}) + i - k - 1.$$

Moreover, since there is no NE-chain of length 3 in F , we have $a_i > a_k$. Thus we deduce that

$$\begin{aligned} x_{i+1} &= \text{asc}(x_1x_2 \dots x_i) + a_i - i \\ &= \text{asc}(x_1x_2 \dots x_{k+1}) + a_i + (i - k - 1) - i \\ &= \text{asc}(x_1x_2 \dots x_{k+1}) + a_i - k - 1 \\ &\geq \text{asc}(x_1x_2 \dots x_{k+1}) + 1 + a_k - k - 1 \\ &= \text{asc}(x_1x_2 \dots x_{k+1}) + a_k - k. \end{aligned}$$

By Lemma 3.3, we have $x_{k+1} = \text{asc}(x_1 x_2 \dots x_{k+1}) + a_k - k$. This yields that $x_{i+1} \geq x_{k+1} \geq 0$. Hence, we have reached the conclusion that $x_1 = 0$ and $0 \leq x_{i+1} \leq \text{asc}(x_1 x_2 \dots x_i) + 1$ for all $i \geq 1$. This ensures that the obtained sequence $\phi'(F)$ is an ascent sequence of length $n + 1$. By Lemma 3.3, we have $x_i \neq x_{i+1}$ for all $i \geq 1$. Hence the obtained sequence $\phi'(F)$ is primitive.

It remains to show that $\phi'(F)$ is 210-avoiding. We claim that for any integers i, j with $i < j$, if $x_{i+1} > x_{j+1}$, then $a_i \geq a_j$. By Lemma 3.3, we have $x_{i+1} = \text{asc}(x_1 x_2 \dots x_{i+1}) + a_i - i$ and $x_{j+1} = \text{asc}(x_1 x_2 \dots x_{j+1}) + a_j - j$. Suppose that $x_{i+1} > x_{j+1}$. It follows that $\text{asc}(x_{i+1} \dots x_{j+1}) \leq j - i - 1$. Then we have

$$\begin{aligned} a_i - a_j &\geq j - i - (\text{asc}(x_1 x_2 \dots x_{j+1}) - \text{asc}(x_1 x_2 \dots x_{i+1})) - 1 \\ &= j - i - \text{asc}(x_{i+1} \dots x_{j+1}) - 1 \\ &\geq 0. \end{aligned}$$

This yields that $a_i \geq a_j$. Hence the claim is proved.

Suppose that $x_{i+1} > x_{j+1} > x_{k+1}$ with $i < j < k$. Then we have $a_i \geq a_j \geq a_k$. This implies that the squares (i, a_i) , (j, a_j) and (k, a_k) form a NE-chain of length 3, which leads to a contradiction. Therefore, the obtained sequence $\phi'(F)$ is 210-avoiding. This completes the proof. \blacksquare

Theorem 3.5 *The maps ϕ and ϕ' are inverses of each other.*

Proof. We first prove that $\phi'(\phi(x)) = x$ for any ascent sequence $x = x_1 x_2 \dots x_{n+1} \in \mathcal{PA}_{n+1}(210)$. Suppose that $\phi(x) = \{(1, a_1), (2, a_2), \dots, (n, a_n)\}$. According to the definition of the map ϕ , we have $a_i = i + x_{i+1} - \text{asc}(x_1 x_2 \dots x_{i+1})$. When we apply the map ϕ' to $\phi(x)$, we get a 210-avoiding ascent sequence $x' = x'_1 x'_2 \dots x'_{n+1}$ such that

- $x'_1 = 0$ and $x'_2 = 1$;
- if $a_{i-1} < a_i$, then $x'_{i+1} = \text{asc}(x'_1 x'_2 \dots x'_i) + 1 + a_i - i$ for all $2 \leq i \leq n$;
- if $a_{i-1} \geq a_i$, then $x'_{i+1} = \text{asc}(x'_1 x'_2 \dots x'_i) + a_i - i$ for all $2 \leq i \leq n$.

We proceed to show that $x'_i = x_i$ for all $i \geq 1$ by induction on i . It is clear that $x'_1 = 0 = x_1$ and $x'_2 = 1 = x_2$. Assume that $x'_k = x_k$ for $1 \leq k \leq i$. Now we proceed to show that $x'_{i+1} = x_{i+1}$. We have two cases.

If $a_{i-1} \geq a_i$, then we have $x'_{i+1} = \text{asc}(x'_1 x'_2 \dots x'_i) + a_i - i$. By the induction hypothesis, we have $\text{asc}(x'_1 x'_2 \dots x'_i) = \text{asc}(x_1 x_2 \dots x_i)$. This implies that $x'_{i+1} = \text{asc}(x_1 x_2 \dots x_i) + a_i - i$. By Lemma 3.1, we have $x_i > x_{i+1}$. This yields

that $asc(x_1x_2 \dots x_{i+1}) = asc(x_1x_2 \dots x_i)$ and $x_{i+1} = asc(x_1x_2 \dots x_{i+1}) + a_i - i$. According to the definition of the map ϕ , we have $a_i = i + x_{i+1} - asc(x_1x_2 \dots x_{i+1})$. Thus we have $x'_{i+1} = x_{i+1}$.

If $a_{i-1} < a_i$, then we have $x'_{i+1} = asc(x'_1x'_2 \dots x'_i) + 1 + a_i - i$. By Lemma 3.3, we have $x'_{i+1} = asc(x'_1x'_2 \dots x'_{i+1}) + a_i - i$ and $x'_i < x'_{i+1}$. This implies that $asc(x'_1x'_2 \dots x'_{i+1}) = asc(x'_1x'_2 \dots x'_i) + 1$. By the induction hypothesis, we have $asc(x'_1x'_2 \dots x'_i) = asc(x_1x_2 \dots x_i)$. It follows that

$$asc(x'_1x'_2 \dots x'_{i+1}) = asc(x'_1x'_2 \dots x'_i) + 1 = asc(x_1x_2 \dots x_i) + 1.$$

By lemma 3.1, we have $x_{i+1} > x_i$. This yields that $asc(x_1x_2 \dots x_{i+1}) = asc(x_1x_2 \dots x_i) + 1$. Thus we deduce that $x'_{i+1} = asc(x_1x_2 \dots x_{i+1}) + a_i - i$. Recall that $a_i = i + x_{i+1} - asc(x_1x_2 \dots x_{i+1})$. Thus we deduce that $x'_{i+1} = x_{i+1}$. Hence we have reached the conclusion that $\phi'(\phi(x)) = x$.

Next we turn to the proof of $\phi(\phi'(F)) = F$ for any 01-filling $F \in \mathcal{N}(\Delta_n)$. Let $F = \{(1, a_1), (2, a_2), \dots, (n, a_n)\}$. Suppose that $\phi'(F) = x_1x_2 \dots x_{n+1}$. When we apply the map ϕ to $\phi'(F)$, we get a 01-filling $F' = \{(1, a'_1), (2, a'_2), \dots, (n, a'_n)\} \in \mathcal{N}(\Delta_n)$. By the definition of the map ϕ and Lemma 3.3, we have $x_{i+1} = asc(x_1x_2 \dots x_{i+1}) + a_i - i$ and $a'_i = i + x_{i+1} - asc(x_1x_2 \dots x_{i+1})$ for $i \geq 1$. This implies that $a'_i = a_i$ for $i \geq 1$. Thus we deduce that $\phi(\phi'(F)) = F$. This completes the proof. \blacksquare

Now we are ready to describe a bijection between the set $\mathcal{A}_{n+1}(210)$ and the set \mathcal{V}_n . Our bijection relies on the growth diagrams for 01-fillings of triangular shape and the bijection ϕ .

Theorem 3.6 *For $n \geq 0$, there is a bijection between the set $\mathcal{A}_{n+1}(210)$ and the set \mathcal{V}_n .*

Proof. Let x be a 210-avoiding ascent sequence of length $n + 1$, we wish to construct a sequence $\psi(x) \in \mathcal{V}_n$. It is apparent that the ascent sequence x can be written as $x_1^{c_1}x_2^{c_2} \dots x_{k+1}^{c_{k+1}}$, where $x_i \neq x_{i+1}$ and $c_i \geq 1$ for all $i \geq 1$. Let $x' = x_1x_2 \dots x_{k+1}$. Obviously, x' is a 210-avoiding primitive ascent sequence of length $k + 1$. For $k = 0$, let $\psi(x)$ be a sequence of length $2n + 1$ consisting of \emptyset 's. For $k \geq 1$, by Theorem 3.2, we obtain a 01-filling $\phi(x')$ in $\mathcal{N}(\Delta_k)$ by applying the map ϕ to x' . When we apply the forward algorithm to $\phi(x')$, we get the growth diagram for the 01-filling $\phi(x')$. Let $(\emptyset = \lambda^0, \lambda^1, \dots, \lambda^{2k} = \emptyset)$ be the sequence of partitions labelling the right/up border of the growth diagram from top-left to bottom-right. By Theorem 2.6, the sequence $(\emptyset = \lambda^0, \lambda^1, \dots, \lambda^{2k} = \emptyset)$ has the properties that the most number of columns of any λ^i is at most 2, and λ^{2i+1} is obtained from λ^{2i} by doing nothing or adding a horizontal strip, whereas λ^{2i+2} is

obtained from λ^{2i} by deleting a square. Now we proceed to generate a sequence of partitions from the sequence $(\emptyset = \lambda^0, \lambda^1, \dots, \lambda^{2k} = \emptyset)$ by the following procedure.

- For all $i \geq 0$, if λ^{2i+1} is obtained from λ^{2i} by adding two squares, then let $\lambda^{2i+1} = \lambda^{2i+2}$. Let $(\mu^0, \mu^1, \dots, \mu^{2k})$ denote the resulting sequence.
- If $c_1 > 1$, then adjoin $2(c_1 - 1)$ copies of μ^0 immediately left to μ^0 .
- For all $1 \leq i \leq k$, if $c_{i+1} > 1$, then insert $2(c_{i+1} - 1)$ copies of μ^{2i} immediately left to μ^{2i} and right to μ^{2i-1} .

Since $c_1 + c_2 + \dots + c_{k+1} = n + 1$, the resulting sequence is of length $2n + 1$. Denote by $(\rho^0, \rho^1, \dots, \rho^{2n})$ the resulting sequence and set $\psi(x) = (\rho^0, \rho^1, \dots, \rho^{2n})$.

In order to show that $\psi(x) \in \mathcal{V}_n$, it suffices to show that the sequence $(\rho^0, \rho^1, \dots, \rho^{2n})$ has the following properties:

- (i) $\rho^0 = \emptyset$ and $\rho^{2n} = \emptyset$;
- (ii) ρ^{2i+1} is obtained from ρ^{2i} by doing nothing or adding a square;
- (iii) ρ^{2i+2} is obtained from ρ^{2i+1} by doing nothing or deleting a square;
- (iv) each ρ^i has at most two columns.

From the construction of the map ψ , it is easily seen that the sequence $(\rho^0, \rho^1, \dots, \rho^{2n})$ has properties (i) and (iv). Moreover, it is easy to check that the sequence $(\rho^0, \rho^1, \dots, \rho^{2n})$ is defined by

$$\rho^l = \begin{cases} \mu^{2i}, & \text{if } \sum_{j=0}^i 2c_j \leq l \leq \sum_{j=0}^{i+1} 2c_j - 2, \\ \mu^{2i+1}, & \text{if } l = \sum_{j=0}^{i+1} 2c_j - 1, \end{cases} \quad (3.1)$$

with the assumption $c_0 = 0$.

We claim that we have either (1) μ^{2i+1} is obtained from μ^{2i} by doing nothing or adding a square and μ^{2i+2} is obtained from μ^{2i+1} by deleting a square, or (2) μ^{2i+1} is obtained from μ^{2i} by adding a square and μ^{2i+2} is obtained from μ^{2i+1} by doing nothing. Since each λ^i has at most two columns, λ^{2i+1} is obtained from λ^{2i} by adding a horizontal strip of at most two squares. If λ^{2i+1} is obtained from λ^{2i} by doing nothing or adding a square, then we have $\mu^{2i} = \lambda^{2i}$, $\mu^{2i+1} = \lambda^{2i+1}$, and $\mu^{2i+2} = \lambda^{2i+2}$. Recall that λ^{2i+2} is obtained from λ^{2i+1} by deleting a square. Thus μ^{2i+1} is obtained from μ^{2i} by doing nothing or adding a square and μ^{2i+2} is obtained from μ^{2i+1} by deleting a square.

If λ^{2i+1} is obtained from λ^{2i} by adding a horizontal strip of two squares, then according to the construction of the map ψ , we have $\mu^{2i+1} = \lambda^{2i+2} = \mu^{2i+2}$. Recall that λ^{2i+2} is obtained from λ^{2i+1} by deleting a square. Hence λ^{2i+2} and λ^{2i} differ by exactly one square. Since $\mu^{2i} = \lambda^{2i}$ and $\mu^{2i+2} = \lambda^{2i+2} = \mu^{2i+1}$, it follows that μ^{2i+1} is obtained from μ^{2i} by adding a square and μ^{2i+2} is obtained from μ^{2i+1} by doing nothing. Hence the claim is proved.

By Formula 3.1, we deduce that either (1) $\rho^{\sum_{j=0}^{i+1} 2c_j - 1}$ is obtained from $\rho^{\sum_{j=0}^{i+1} 2c_j - 2}$ by doing nothing or adding a square and $\rho^{\sum_{j=0}^{i+1} 2c_j}$ is obtained from $\rho^{\sum_{j=0}^{i+1} 2c_j - 1}$ by deleting a square, or (2) $\rho^{\sum_{j=0}^{i+1} 2c_j - 1}$ is obtained from $\rho^{\sum_{j=0}^{i+1} 2c_j - 2}$ by adding a square and $\rho^{\sum_{j=0}^{i+1} 2c_j}$ is obtained from $\rho^{\sum_{j=0}^{i+1} 2c_j - 1}$ by doing nothing. Moreover, for all $\sum_{j=0}^i 2c_j < l \leq \sum_{j=0}^{i+1} 2c_j - 2$, we have $\rho^l = \rho^{l-1}$. Hence, the sequence $(\rho^0, \rho^1, \dots, \rho^{2n})$ has properties (ii) and (iii), which implies that $\psi(x) \in \mathcal{V}_n$.

Conversely, given a sequence $V = (\rho^0, \rho^1, \dots, \rho^{2n}) \in \mathcal{V}_n$, we wish to recover a 210-avoiding ascent sequence $\psi'(V)$. If $\rho^i = \emptyset$ for all $i \geq 0$, then let $\psi'(V)$ be the ascent sequence of length $n+1$ consisting of 0's. Otherwise, for all $i \geq 0$, remove ρ^{2i} and ρ^{2i+1} from the sequence $(\rho^0, \rho^1, \dots, \rho^{2n})$ whenever $\rho^{2i} = \rho^{2i+1} = \rho^{2i+2}$. Assume that

$$(\rho^{2j_0}, \rho^{2j_0+1}, \rho^{2j_1}, \rho^{2j_1+1}, \dots, \rho^{2j_{m-1}}, \rho^{2j_{m-1}+1}, \rho^{2j_m})$$

is the resulting sequence. Let $(v^0, v^1, \dots, v^{2m})$ be a sequence such that (1) $v^{2i} = \rho^{2j_i}$; (2) if $\rho^{2j_i+1} = \rho^{2j_{i+1}}$, then v^{2i+1} is a partition obtained from v^{2i} by adding a horizontal strip of two squares; (3) otherwise, $v^{2i+1} = \rho^{2j_{i+1}}$. By applying the backward algorithm to $(v^0, v^1, \dots, v^{2m})$, we obtain a 01-filling $F \in \mathcal{N}(\Delta_m)$. Let $\phi'(F) = y_1 y_2 \dots y_{m+1}$ and $\psi'(V) = y_1^{c'_1} y_2^{c'_2} \dots y_{m+1}^{c'_{m+1}}$, where $c'_{i+1} = j_i - j_{i-1}$ for $i \geq 1$ and $c'_1 = j_0 + 1$.

It is apparent that we have $j_m = n$. Since $c'_1 + c'_2 + \dots + c'_{m+1} = n+1$, the obtained sequence $\psi'(V)$ is a sequence of length $n+1$. By Theorems 2.6 and 3.4, in order to show that $\psi'(V) \in \mathcal{A}_{n+1}(210)$, it suffices to show that the sequence $(v^0, v^1, \dots, v^{2m})$ verifies the following points.

- (i)' $v^0 = v^{2m} = \emptyset$;
- (ii)' v^{2i+1} is obtained from v^{2i} by doing nothing or adding a horizontal strip;
- (iii)' v^{2i+2} is obtained from v^{2i+1} by deleting a square;
- (iv)' each v^i has at most two columns.

It is easy to check that the statements of (i)', (ii)' and (iv)' are true for the sequence $(v^0, v^1, \dots, v^{2m})$. From the construction of the map ψ' , it is easily seen

that if $\rho^{2j_i+1} \neq \rho^{2j_i+1}$, then $v^{2i+2} = \rho^{2j_i+1}$ and $v^{2i+1} = \rho^{2j_i+1}$. Moreover, we have $\rho^{2j_i+2} = \rho^{2j_i+1}$. This implies that ρ^{2j_i+2} is obtained from ρ^{2j_i+1} by deleting a square. Hence, it follows that v^{2i+2} is obtained from v^{2i+1} by deleting a square.

In order to verify (iii)', it remains to show that if $\rho^{2j_i+1} = \rho^{2j_i+1}$, then v^{2i+2} is obtained from v^{2i+1} by deleting a square. Suppose that $\rho^{2j_i+1} = \rho^{2j_i+1}$. From the construction of the sequence

$$(\rho^{2j_0}, \rho^{2j_0+1}, \rho^{2j_1}, \rho^{2j_1+1}, \dots, \rho^{2j_{m-1}}, \rho^{2j_{m-1}+1}, \rho^{2j_m}),$$

it follows that $\rho^{2j_i+1} \neq \rho^{2j_i}$. This implies that ρ^{2j_i+1} is obtained from ρ^{2j_i} by adding one square. Recall that $v^{2i} = \rho^{2j_i}$. This implies that v^{2i+1} is obtained from v^{2i} by adding a horizontal strip of two squares. Since $v^{2i+2} = \rho^{2j_i+1} = \rho^{2j_i+1}$ and ρ^{2j_i+1} is obtained from ρ^{2j_i} by adding one square, the partition v^{2i+2} is obtained from v^{2i} by adding one square. Recall that v^{2i+1} is obtained from v^{2i} by adding a horizontal strip of two squares. It follows that v^{2i+2} is obtained from v^{2i+1} by deleting a square. Hence, we have reached the conclusion that $\psi'(V) \in \mathcal{A}_{n+1}(210)$.

Now we proceed to show that the map ψ is indeed a bijection. To this end, we will show that the maps ψ and ψ' are inverses of each other. We first show that the map ψ' is the inverse of the map ψ , that is, $\psi'(\psi(x)) = x$. By Theorems 3.5 and 2.6, it suffices to prove that

$$(\rho^{2j_0}, \rho^{2j_0+1}, \rho^{2j_1}, \rho^{2j_1+1}, \dots, \rho^{2j_{m-1}}, \rho^{2j_{m-1}+1}, \rho^{2j_m}) = (\mu^0, \mu^1, \dots, \mu^{2k}), \quad (3.2)$$

and $c_j = c'_j$ for all $j \geq 1$. Recall that we have either (1) $\rho^{\sum_{j=0}^{i+1} 2c_j-1}$ is obtained from $\rho^{\sum_{j=0}^{i+1} 2c_j-2}$ by doing nothing or adding a square and $\rho^{\sum_{j=0}^{i+1} 2c_j}$ is obtained from $\rho^{\sum_{j=0}^{i+1} 2c_j-1}$ by deleting a square, or (2) $\rho^{\sum_{j=0}^{i+1} 2c_j-1}$ is obtained from $\rho^{\sum_{j=0}^{i+1} 2c_j-2}$ by adding a square and $\rho^{\sum_{j=0}^{i+1} 2c_j}$ is obtained from $\rho^{\sum_{j=0}^{i+1} 2c_j-1}$ by doing nothing. Moreover, for all $\sum_{j=0}^i 2c_j < l \leq \sum_{j=0}^{i+1} 2c_j - 2$, we have $\rho^l = \rho^{l-1}$. From the construction of the map ψ' , it follows that $j_i = \sum_{j=0}^{i+1} c_j - 1$ for all $i \geq 0$. This implies that $c'_1 = j_0 + 1 = c_1$ and $c'_{i+1} = j_i - j_{i-1} = c_{i+1}$ for all $i \geq 1$. Moreover, by Formula 3.1, we have $\rho^{\sum_{j=0}^{i+1} 2c_j-2} = \mu^{2i}$ and $\rho^{\sum_{j=0}^{i+1} 2c_j-1} = \mu^{2i+1}$. Thus we obtain Formula 3.2.

By Theorems 3.5 and 2.6, it is routine to check that the map ψ reverses each step of the map ψ' . This implies that the map ψ is the inverse map of ψ' . Hence, the maps ψ and ψ' are inverses of each other. This completes the proof. \blacksquare

For example, let $x = 0012303222353$ be a 210-avoiding ascent sequence of length 13. Then x can be written as $0^2 1^1 2^1 3^1 0^1 3^1 2^3 3^1 5^1 3^1$. Let $x' = 0123032353$, which is a 210-avoiding primitive ascent sequence of length 10. By applying the

map ϕ to x' , we get a 01-filling

$$\phi(x') = \{(1, 1), (2, 2), (3, 3), (4, 1), (5, 4), (6, 4), (7, 5), (8, 7), (9, 6)\} \in \mathcal{N}(\Delta_9).$$

Then we get the growth diagram for the 01-filling $\psi(x')$ illustrated in Figure 6. Let

$$(\lambda^0, \lambda^1, \dots, \lambda^{18}) = (\emptyset, 2, 1, 2, 1, 2, 1, 21, 2, 21, 11, 111, 11, 21, 2, 2, 1, 1, \emptyset)$$

be the sequence of partitions labelling the corners along the right/up border of the growth diagram from top-left to bottom-right. It is easy to check that λ^1 is obtained from λ^0 by adding two squares and λ^7 is obtained from λ^6 by adding two squares. Hence we get a sequence

$$(\mu^0, \mu^1, \dots, \mu^{18}) = (\emptyset, 1, 1, 2, 1, 2, 1, 2, 2, 21, 11, 111, 11, 21, 2, 2, 1, 1, \emptyset)$$

by replacing λ^1 and λ^7 with the partitions $\lambda^2 = 1$ and $\lambda^8 = 2$, respectively. Finally, we obtain a sequence

$$\psi(x) = (\emptyset, \emptyset, \emptyset, 1, 1, 2, 1, 2, 1, 2, 2, 21, 11, 111, 11, 11, 11, 11, 11, 21, 2, 2, 1, 1, \emptyset) \in \mathcal{V}_{12}$$

from $(\mu^0, \mu^1, \dots, \mu^{18})$ by inserting two copies of μ^0 immediately left to μ^0 , and inserting four copies of μ^{12} immediately left to μ^{12} and right to μ^{11} .

Conversely, given a sequence of partitions

$$V = (\rho^0, \rho^1, \dots, \rho^{24}) = (\emptyset, \emptyset, \emptyset, 1, 1, 2, 1, 2, 1, 2, 2, 21, 11, 111, 11, 11, 11, 11, 11, 21, 2, 2, 1, 1, \emptyset),$$

we aim to recover an ascent sequence $\psi'(V)$. It is easily seen that $\rho^0 = \rho^1 = \rho^2$, $\rho^{14} = \rho^{15} = \rho^{16}$ and $\rho^{16} = \rho^{17} = \rho^{18}$. We obtain a sequence

$$(\rho^2, \rho^3, \rho^4, \rho^5, \rho^6, \rho^7, \rho^8, \rho^9, \rho^{10}, \rho^{11}, \rho^{12}, \rho^{13}, \rho^{18}, \rho^{19}, \rho^{20}, \rho^{21}, \rho^{22}, \rho^{23}, \rho^{24})$$

by removing $\rho^0, \rho^1, \rho^{14}, \rho^{15}, \rho^{16}, \rho^{17}$. Moreover, we have $\rho^3 = \rho^4$ and $\rho^9 = \rho^{10}$. Replace ρ^3 and ρ^9 with the partitions 2 and 21 respectively. This leads to a sequence of partitions

$$(v^0, v^1, \dots, v^{18}) = (\emptyset, 2, 1, 2, 1, 2, 1, 21, 2, 21, 11, 111, 11, 21, 2, 2, 1, 1, \emptyset).$$

Applying the backward algorithm, we get a 01-filling

$$F = \{(1, 1), (2, 2), (3, 3), (4, 1), (5, 4), (6, 4), (7, 5), (8, 7), (9, 6)\}.$$

Finally we have $\phi'(F) = 0123032353$ and $\psi'(V) = 0012303222353$.

Combining Theorems 2.7, 2.8 and 3.6, we get the following theorem, which leads to a combinatorial proof of Conjecture 1.1.

Theorem 3.7 *210-avoiding ascent sequences of length n are in bijection with 3-nonnesting set partitions of $[n]$.*

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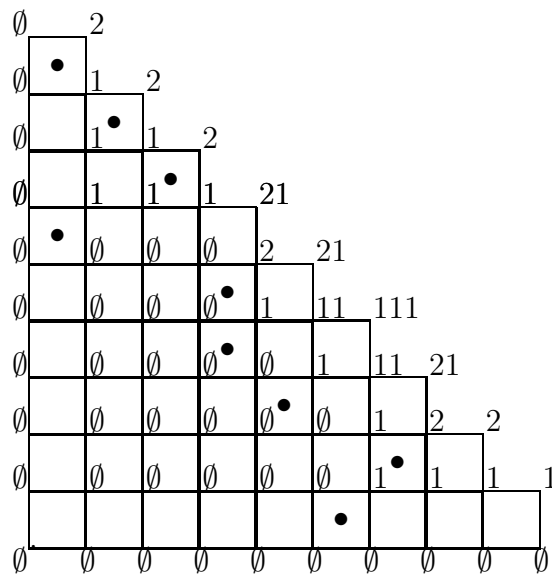


Figure 6: The growth diagram for a 01-filling of Δ_9 .

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